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NOTE ON THE THEORY OF FUNCTIONS.

By Prof. W. H. Echols, Charlottesville, Va.

1. While the Theory of Functions of a Complex Variable, together with all the results, expansions, etc. of that theory apply in general to the case when the variable is assumed to be real, care must be taken in this particularization of the variable in order to give the proper interpretation to the results, which at first sight may not be evident. This is one of the difficulties which presents itself to the student on passing from the real to the complex. He is too apt to feel that between the region of the real and that of the imaginary, there lies an uncertain ground in which the danger signals to be avoided are too frequently displayed. This field of apparent separation is too humble a one for cultivation by the master analysts to whose minds no difficulty presents itself, and who therefore deem the field a barren one. But few are endowed with the creative inspiration and can serve as pioneers in exploratory investigation, and the work of file-closing seems to me to be at the present stage of mathematical work not less important.

2. The foundation of Cauchy's theory of the complex function reposes on the theorem resulting from the integration, around a complete boundary, of a holomorphic function having within the boundary a single pole. In symbols this theorem is expressed by

$$\int_{(C)} \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)\,,$$

in which $f(z)$ is a function of $z = x + iy$, which is holomorphic throughout the area including the boundary determined by C, and a is a point of this area, i being $+\sqrt{-1}$. If we place $y = 0$, we assume the variable to be real. What then is the corresponding interpretation which must be given to Cauchy's theorem? It is the object of this note to attempt the answer to this question, and to interpret some of the simpler theorems which flow from it. In order to essay this, we approach the problem wholly from the domain of real quantity, remarking subsequently upon the correlation between the real and the complex.

3. Let $f(x)$ be a uniform, finite, and continuous function of the real variable x throughout the interval from $x = \alpha$ to $x = \beta$, inclusive. We call the integral

$$\int_{(\alpha\beta)} f(x)\, dx\,,$$

in which the integral is taken from a to β and then back to a again, the *complete integral* of the function throughout the interval $(a\beta)$. This integral, when $f(x)$ is uniform, finite, and continuous throughout $(a\beta)$, is evidently zero. Consider the complete integral

$$\int_{(a\beta)} \frac{f(x)}{x-a}\,dx\,,$$

in which $a < a < \beta$. Let ε be a small *positive* quantity. We have

$$\int_{(a\beta)} \frac{f(x)}{x-a}\,dx = \int_{a}^{a-\epsilon} \frac{f(x)}{x-a}\,dx + \int_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx + \int_{a+\epsilon}^{\beta} \frac{f(x)}{x-a}\,dx$$

$$+ \int_{\beta}^{a+\epsilon} \frac{f(x)}{x-a}\,dx + \int_{a+\epsilon}^{a-\epsilon} \frac{f(x)}{x-a}\,dx + \int_{a-\epsilon}^{a} \frac{f(x)}{x-a}\,dx\,.$$

The function $f(x)/(x-a)$ being uniform, finite, and continuous throughout the intervals $(a, a-\varepsilon)$ and $(a+\varepsilon, \beta)$, the complete integral evidently becomes

$$\int_{(a\beta)} \frac{f(x)}{x-a}\,dx = \int_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx + \int_{a+\epsilon}^{a-\epsilon} \frac{f(x)}{x-a}\,dx\,,$$

and therefore has no value unless it be in the immediate neighborhood of the pole a. The function $f(x)/(x-a)$ being infinite at a, the above integrals require special consideration in the interval $(a-\varepsilon, a+\varepsilon)$. We have

$$\int_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx = \int_{a-\epsilon}^{a+\epsilon} \frac{f(x)-f(a)}{x-a}\,dx + \int_{a-\epsilon}^{a+\epsilon} \frac{f(a)}{x-a}\,dx$$

$$= \int_{a-\epsilon}^{a} \frac{f(x)-f(a)}{x-a}\,dx + \int_{a}^{a+\epsilon} \frac{f(x)-f(a)}{x-a}\,dx + f(a)\int_{a-\epsilon}^{a+\epsilon} \frac{dx}{x-a}\,.$$

If the function $f(x)$ has a determinate finite derivative throughout the interval $(a-\varepsilon, a+\varepsilon)$, we have

$$f(x) - f(a) = (x-a)\,f'(u)\,,$$

where u is some value of x in this interval between x and a. The integral is therefore

$$\int_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx = \int_{a-\epsilon}^{a} f'(u)\,dx + \int_{a}^{a+\epsilon} f'(u)\,dx + f(a)\,.\log(-1)\,.$$

Now let ε diminish indefinitely, and we have in the limit

$$\int_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx = f(a)\,.\log(-1)\,,$$

since the first two integrals on the right evidently vanish.

No real quantity can express the logarithm of an essentially negative real quantity. We therefore recognize in log (— 1) an unreal form. Is the result nugatory? Shall we, assuming no further knowledge of the nature of the symbol log (— 1), consider it to be an unreal indeterminate absurdity and discontinue the investigation, or conceive it to be a determinate analytical constant subject to the fundamental laws of algebra and examine the consequences which flow from its use as such? Choosing the latter course, let us represent log (— 1) by the symbol j. We then have, for the complete integral

$$\int_{(a\beta)} \frac{f(x)}{x-a}\,dx = 2j\,f(a)\,.$$

The integral having no appreciable value except in the neighborhood of a, we may conventionally say, the value of the complete integral throughout the interval $(a\beta)$ is its value at the pole a, and write

$$\int_{(a\beta)} \frac{f(x)}{x-a}\,dx = \int_{(a)} \frac{f(x)}{x-a}\,dx = 2j\,f(a)\,.$$

4. Cauchy's Theory of Functions of a Complex Variable is based on the introduction into analysis of the imaginary unit or operator $i = +\sqrt{-1}$, which is taken as the unit of imaginary metrical length. This is but a conventional selection, arbitrarily made, of one particular value (the simplest) of the general symbol

$$(\pm 1)^{2n\pm 1}\sqrt{(-1)^{2n\pm 1}}\,.$$

We conventionally represent by i the analytical constant derived by making $n = 0$ and using only the upper sign. In this Theory, we have

$$\log(a + ib) = \tfrac{1}{2}\log(a^2 + b^2) + i\tan^{-1}(b/a)\,,$$

which for $b = 0$ and a essentially real and negative, we have

$$\log(-1) = i\tan^{-1}(-0) = i(2n \pm 1)\pi\,,$$

or, log (— 1) is an imaginary angle, which we may take as the unit of imaginary angles, by selecting that particular value of the general symbol

$$\log(-1) = (2n \pm 1)\pi\,(\pm 1)^{2n\pm 1}\sqrt{(-1)^{2n\pm 1}}\,,$$

under the same convention which selects the analytical constant unit i, by making $n = 0$ and using only the upper sign, which gives

$$\log(-1) = i\pi = j\,.$$

It is obvious that no integer value of n can alter the value of $\log(-1)$ so far as absolute position angle is concerned. There is in this sense no more indetermination in the general value of $\log(-1)$ than there is in the general value of i. Any imaginary angle expressed in unit angles is therefore

$$\theta \log(-1) = j\theta\,,$$

θ being a real number. If we choose to give geometrical interpretation to j as an operator, then, θ being the number which measures a real angle with respect to an initial line, the effect of the operator j on θ is to turn the plane of the real angle about its initial line through a right angle.

5. The theorems which flow from the integral at the end of § 3, of course follow in the same manner as do the generalizations of Cauchy's formula for a complex variable. For example, differentiating

$$f(a) = \frac{1}{2j}\int_{(\alpha\beta)} \frac{f(x)}{x-a}\,dx \tag{1}$$

with respect to a, gives

$$\frac{f^n(a)}{n\,!} = \frac{1}{2j}\int_{(\alpha\beta)} \frac{f(x)}{(x-a)^{n+1}}\,dx\,. \tag{2}$$

Again, let

$$\Delta^{nh} f(a) = f(a + nh) - C_{n,1} f(a + \overline{n-1}\,h) + \ldots + (-1)^n f(a)$$

be the nth progressive difference* of $f(x)$ at a, h being the scale of difference, and the points $(a + rh)$ $(r = 0, \ldots, n)$ points in $(\alpha\beta)$. Then in virtue of (1), we easily find

$$2j\,\Delta^{nh} f(a) = \int_{(\alpha\beta)} f(x)\,dx\,.\,\Delta^{n-h}\frac{(-1)^n}{(x-a)^{1h}}\,;$$

whence

$$\frac{\Delta^{nh} f(a)}{n\,!\,h^n} = \frac{1}{2j}\int_{(\alpha\beta)} \frac{f(x)}{(x-a)^{n+1-h}}\,dx\,, \tag{3}$$

wherein $(x-a)^{n+1-h} = (x-a)(x-a-h)\ldots(x-a-nh)\,.$

* See Note on the Theory of Functions of a Real Variable. ANNALS OF MATHEMATICS, Vol. VIII, pp. 66, 67.

In like manner, if the points $(a - rh)$ $(r = 0, \ldots, n)$ are in $(\alpha\beta)$, we have for the regressive difference-ratio at a

$$\frac{\Delta^{n-h} f(a)}{n!\, h^n} = \frac{1}{2j} \int_{(\alpha\beta)} \frac{f(x)}{(x-a)^{n+1+h}}\, dx\,. \tag{4}$$

If the nth derivative of $f(x)$ is finite and determinate, making $h = 0$ in (3) or (4) we derive (2).

6. Consider the complete integral

$$\int_{(\alpha\beta)} \frac{f(t)\, dt}{(t - a_0) \ldots (t - a_n)},$$

wherein $a_0, \ldots, a_n$ are any points in $(\alpha\beta)$, and t the running point in the interval. We have by the ordinary rule for expanding rational fractions

$$\frac{1}{(t - a_0) \ldots (t - a_n)} = \sum_0^n{}_r \frac{1}{(t - a_r)\, \varphi'(a_r)},$$

wherein

$$\varphi(t) = (t - a_0) \ldots (t - a_n),$$

and $\varphi'(a_r)$ is the derivative of $\varphi(t)$ with respect to t and t is replaced by a_r.

Therefore by the result of (1), we have*

$$\begin{aligned}
\frac{1}{2j} \int_{(\alpha\beta)} \frac{f(t)\, dt}{(t - a_0) \ldots (t - a_n)} &= \sum_0^n{}_r \frac{f(a_r)}{\varphi'(a_r)} \\
&= \sum_0^n{}_r \frac{f(a_r)}{(a_r - a_0) \ldots (a_r - a_{r-1})(a_r - a_{r+1}) \ldots (a_r - a_n)} \\
&= \sum_0^n{}_r (-1)^r \frac{\zeta^{\frac12}(a_0, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n)}{\zeta^{\frac12}(a_0, \ldots, a_n)} f(a_r) \\
&= \frac{\begin{vmatrix} f(a_0) & 1 & a_0 & \ldots & a_0^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \ldots & a_n^{n-1} \end{vmatrix}}{\zeta^{\frac12}(a_0, \ldots, a_n)}\,.
\end{aligned}$$

Otherwise, and very much more briefly, we need only observe that the complete integral through $(\alpha\beta)$ is but the sum of the complete integrals at the poles, and write

$$\int_{(\alpha\beta)} \frac{f(t)\, dt}{\varphi(t)} = \int_{(a_0)} \frac{f(t)\, dt}{\varphi(t)} + \ldots + \int_{(a_n)} \frac{f(t)\, dt}{\varphi(t)} = \sum_0^n{}_r \frac{f(a_r)}{\varphi'(a_r)}\,.$$

* Muir's Determinants, p. 165.

Consider the function

$$\frac{\begin{vmatrix} f(x) & 1 & x & \dots & x^n \\ f(a_1) & 1 & a_1 & \dots & a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \dots & a_n^n \\ f(a_0) & 1 & a_0 & \dots & a_0^n \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_0, \dots, a_n)},$$

which vanishes at the points $a_0, \dots, a_n$. Its nth derivative vanishes at the point u between the greatest and least of the values $a_0, \dots, a_n$, and we have

$$\frac{f^n(u)}{n!} = \frac{\begin{vmatrix} f(a_0) & 1 & a_0 & \dots & a_0^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \dots & a_n^{n-1} \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_0, \dots, a_n)},$$

$$= \frac{1}{2j} \int_{(\alpha\beta)} \frac{f(t)\, dt}{(t - a_0) \dots (t - a_n)}.$$

If in this result we let $a_0 = a_1 = \dots = a_n = a$, we get the expression (2), since then u is also equal to a. We notice that the ratio of the determinants above takes the form $0/0$ when the a's become a. If we consider this we have

$$\frac{1}{2j} \int_{(\alpha\beta)} \frac{f(t)\, dt}{(t - x)(t - a_0) \dots (t - a_n)} = \frac{f^{n+1}(u)}{(n + 1)!}$$

$$= \frac{\begin{vmatrix} f(x) & 1 & x & \dots & x^n \\ f(a_0) & 1 & a_0 & \dots & a_0^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \dots & a_n^n \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, a_0, \dots, a_n)}.$$

In order to remove the indetermination in this last expression caused by the a's becoming equal to a, we operate on the numerator and denominator with

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1 = a} \dots \left[\frac{\partial}{\partial a_n}\right]^n_{a_n = a},$$

and after simplification and expansion, we have*

$$\frac{1}{2j}\int_{(\alpha\beta)}\frac{f(t)\,dt}{(t-x)(t-a)^{n+1}}=\frac{f^{n+1}(u)}{(n+1)!} \tag{5}$$

$$=\frac{1}{(x-a)^{n+1}}\left\{f(x)-f(a)-(x-a)f'(a)-\ldots-\frac{(x-a)^n}{n!}f^n(a)\right\},$$

wherein u lies between x and a.

Another neat way of getting this result is to consider (5) in the form

$$\int_{(\alpha\beta)}\frac{\dfrac{f(x)-f(t)}{x-t}}{(t-a)^{n+1}}\,dt=\frac{f^{n+1}(u)}{(n+1)!}.$$

The integral on the left is, by (2), equal to

$$\frac{2j}{n!}\left[\frac{d}{dt}\right]^n_{t=a}\left[\frac{f(x)-f(t)}{x-t}\right],$$

since the function

$$\frac{f(x)-f(t)}{x-t}$$

has no pole in $(\alpha\beta)$. But

$$\left[\frac{d}{dt}\right]^n_{t=a}\left[\frac{f(x)-f(t)}{x-t}\right]=\frac{n!}{(x-a)^{n+1}}\left\{f(x)-f(a)-(x-a)f'(a)-\ldots-\frac{(x-a)^n}{n!}f^n(a)\right\}.$$

Whence at once

$$f(x)=f(a)+(x-a)f'(a)+\ldots+\frac{(x-a)^n}{n!}f^n(a)+\frac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(u).$$

Again, obviously, the integral in (5) can have no appreciable value except in the neighborhood of the points x and a. We may therefore write it

$$\int_{(\alpha\beta)}\frac{f(t)\,dt}{(t-x)(t-a)^{n+1}}=\int_{(x)}\frac{(t-a)^{-n-1}f(t)}{t-x}\,dt+\int_{(a)}\frac{(t-x)^{-1}f(t)}{(t-a)^{n+1}}\,dt.$$

But

$$\int_{(x)}\frac{(t-a)^{-n-1}f(t)}{t-x}\,dt=2j\left[\frac{f(t)}{(t-a)^{n+1}}\right]_{t=x}=2j\frac{f(x)}{(x-a)^{n+1}}.$$

* See Note on Theory of Functions of a Real Variable, ANNALS OF MATHEMATICS, Vol. VIII, pp. 65, *et seq.*

Also,

$$\int_{(\alpha\beta)} \frac{(t-x)^{-1} f(t)}{(t-a)^{n+1}} dt = \frac{2j}{n!} \left[\frac{\partial}{\partial t}\right]^n_{t=a} \left[\frac{f(t)}{t-x}\right],$$

and

$$\frac{(x-a)^{n+1}}{n!} \left[\frac{\partial}{\partial t}\right]^n_{t=a} \left[\frac{f(t)}{x-t}\right] = f(a) + (x-a) f'(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a),$$

which is the same result as above.

7. If we apply the formula (1) directly to the expansion of functions in series, as is done in the Theory of Holomorphic Functions of a Complex Variable, we have

$$2j f(x) = \int_{(\alpha\beta)} \frac{f(t)}{t-x} dt,$$

x being a point in $(\alpha\beta)$. We also have the identity

$$\frac{1}{t-x} = \frac{1}{(t-a)\left[1 - \dfrac{x-a}{t-a}\right]}$$

$$= \frac{1}{t-a} + \frac{x-a}{(t-a)^2} + \ldots + \frac{(x-a)^n}{(t-a)^{n+1}} + \frac{(x-a)^{n+1}}{(t-x)(t-a)^{n+1}}. \qquad (6)$$

Let a be also a point in $(\alpha\beta)$; then, by substitution and application of (2), we have

$$f(x) = f(a) + (x-a) f'(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a)$$

$$+ \frac{(x-a)^{n+1}}{2j} \int_{(\alpha\beta)} \frac{f(t)}{(t-x)(t-a)^{n+1}} dt. \qquad (7)$$

But we have proved this last integral to be

$$2j \frac{f^{n+1}(u)}{(n+1)!},$$

u lying between x and a.

We notice that if we substitute for $(t-x)^{-1}$ its identical value obtained from

$$-\frac{1}{t-x} = \frac{1}{x-a} + \frac{t-x}{(x-a)^2} + \ldots + \frac{(t-x)^n}{(x-a)^{n+1}} + \frac{(t-a)^{n+1}}{(x-a)^{n+1}(x-t)},$$

instead of getting a form equivalent to Laurent's expansion we are led to the

identity

$$\int_{(\alpha\beta)}^{\bullet} \frac{f(t)}{t-x}\,dt = -\frac{1}{(x-a)^{n+1}} \int_{(\alpha\beta)}^{\bullet} \frac{(t-a)^{n+1} f(t)}{t-x}\,dt$$

$$= -\frac{2j}{(x-a)^{n+1}} \left[(t-a)^{n+1} f(t) \right]_{t=x}$$

$$= 2j\,f(x)\,,$$

the integrals of the other terms having vanished because the functions under the $\int$ signs have no pole in $(\alpha\beta)$.

8. Taking for granted that the *complete* integral may be applied with confidence to real functions of a real variable, we go further and apply it to a more general case. Consider the identity*

$$\frac{t}{t-x} = 1 + \frac{x}{t-a_1} + \ldots + \frac{x(x-a_1)\ldots(x-a_{n-1})}{(t-a_1)\ldots(t-a_n)}$$

$$+ \frac{x(x-a_1)\ldots(x-a_n)}{(t-x)(t-a_1)\ldots(t-a_n)}\,.$$

The series on the right is convergent when extended to infinity, and has for its limit $t/(t-x)$, when the quantities involved are such that

$$\prod_{1}^{\infty}{}_n \frac{x+a_n}{t+a_n} = 0\,,$$

which is true when

$$(x-t)\sum_{1}^{\infty}{}_n \frac{a_n}{t+a_n}$$

is infinite and negative.

Let $t = t - a_0$ and $x = x - a_0$. The identity may then be written

$$\frac{1}{t-x} = \frac{1}{t-a_0} + \sum_{1}^{n}{}_r \prod_{0}^{r}{}_p \frac{x-a_{p-1}}{t-a_p} + \prod_{0}^{n}{}_p \frac{x-a_p}{(t-x)(t-a_p)}\,. \tag{8}$$

Now let $x, a_0, \ldots, a_n$ be any points in the interval $(\alpha\beta)$. If we multiply both sides of this identity by $f(t)\,dt$ and integrate completely over the interval $(\alpha\beta)$, we have

$$2j\,f(x) = 2j\,f(a_0) + \int_{(\alpha\beta)}^{\bullet} \sum_{1}^{n}{}_r \prod_{0}^{n}{}_p \frac{x-a_{p-1}}{t-a_p} f(t)\,dt + \int_{(\alpha\beta)}^{\bullet} \prod_{0}^{n}{}_p \frac{x-a_p}{t-a_p} \frac{f(t)}{t-x}\,dt\,. \tag{9}$$

* Derived from Euler's identity. See Chrystal's Algebra II. 393; also, ANNALS OF MATHEMATICS, Vol. VIII, No. 2. Note on Theory of Functions of a Complex Variable.

But we have shown that

$$\int_{(\alpha\beta)} \prod_0^r{}_p \frac{x - a_{p-1}}{t - a_p} f(t)\, dt = 2j \prod_0^r{}_p (x - a_{p-1}) \frac{f^r(u_r)}{r\,!}, \tag{10}$$

wherein u_r is some point in $(\alpha\beta)$ lying between the greatest and least of $a_0, \ldots, a_n$; also that

$$\int_{(\alpha\beta)} \prod_0^n{}_p \frac{x - a_p}{t - a_p} \frac{f(t)}{t - x}\, dt = 2j \prod_0^n{}_p (x - a_p) \frac{f^{n+1}(u)}{(n+1)\,!}, \tag{11}$$

wherein u lies between the greatest and least of $x, a_0, \ldots, a_n$. We may therefore write this equality thus:

$$f(x) = f(a_0) + (x - a_0) f'(u_1) + \ldots + \prod_0^n{}_p (x - a_{n-1}) \frac{f^n(u_n)}{n\,!}$$
$$+ \prod_0^n{}_p (x - a_p) \frac{f^{n+1}(u)}{(n+1)\,!}.$$

It follows from this that whenever $f(x)$ can be expanded in an infinite series of powers of $x - a$ in the interval $(\alpha\beta)$, it can be expanded in an infinite series of the rational integral functions $(x - a_0) \ldots (x - a_r)$ $(r = 0, 1, 2, \ldots)$. Of course Taylor's series is the particular case when $a_0, \ldots, a_n$ all become a. For then u_r $(r = 1, \ldots, n)$ also become a, while u lies between x and a. (9) is then an interpolation formula; and we may suppose all the a's equal after the mth, say equal to a. Then we get the expansion of $f(x)$ in terms of the function at $a_0, \ldots, a_m$, and a and the successive derivatives at a.

In particular, let $a_0 = a$ and $a_p = ph$ $(p = 1, \ldots, n)$, and write

$$(t - a)^{r|-h} = (t - a)(t - a - h) \ldots (t - a - \overline{r - 1}\, h).$$

Then

$$\int_{(\alpha\beta)} \prod_0^r{}_p \frac{x - a_{p-1}}{t - a_p} f(t)\, dt = \int_{(\alpha\beta)} \frac{(x - a)^{r|-h}}{(t - a)^{r+1|-h}} f(t)\, dt$$
$$= 2j \frac{(x - a)^{r|-h}}{r\,!} \frac{\Delta^r f(a)}{h^r},$$

and we have

$$f(x) = f(a) + \frac{(x - a)^{1|-h}}{1\,!} \frac{\Delta f(a)}{h} + \ldots + \frac{(x - a)^{n|-h}}{n\,!} \frac{\Delta^n f(a)}{h^n}$$
$$+ \frac{(x - a)^{n+1|-h}}{(n - 1)\,!} f^{n+1}(u), \tag{12}$$

wherein u lies between x, a, and $a + nh$. If as n increases, we let h decrease

so that $a + nh$ always remains in (a, β), then this series passes into Taylor's for $n = \infty$.

9. The interpretation of § 4, which makes $\log(-1) = j = i\pi$ in particular, makes the foregoing results particularizations of the corresponding theorems in the theory of complex functions. The shrinkage of the boundary of integration to a double line causes the variable to pass through the poles and the argument of the function under consideration during the process of integration. This fact destroys the usefulness of the complete integral as a test of the disappearance of the remainder in series, and throws us back again, as it should, upon the test of the nth derivative. The results which flow from the primal assumption are consistent with the results of the theory of functions of a real variable in every particular; they are consistent also with the results of the theory of functions of a complex variable under the particularization of the variable, and serve to illustrate the transition between the two, besides pointing out, in a manner which I have not seen done otherwise, the advantages of the latter as the more perfect system of analysis. For myself, I at present fail altogether to discern wherein the process is at variance with rigorous analysis of uniform functions.

November, 1893.

HOMOGENEOUS STRAINS.

By Dr. William H. Metzler, Boston, Mass.

Introduction.

A matrix considered as a linear vector operator, when geometrically interpreted, represents a homogeneous strain,* so that the latter subject may be treated by means of matrices. This subject has been very ably and fully treated by Professors Tait and Kelland in their treatises on Quaternions, and I do not here in any way extend their investigations but simply cover some of the same ground making use of the operator φ in the form of a square array and thus exhibiting the roles which its constituents play.

I shall separate the paper into two parts, the first dealing with homogeneous strains in space of two dimensions, and the second part dealing with homogeneous strains in space of three dimensions.

Part I.—Space of Two Dimensions.

1. Let the matrix $\varphi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

and let $\rho = xi + yj$ be the vector to a point whose coordinates are (x, y) referred to the rectangular system i, j; then

$$\varphi\rho = (a_{11}x + a_{12}y)\,i + (a_{21}x + a_{22}y)\,j = \rho' .$$

The angle θ between ρ and $\varphi\rho$ is given by

$$\cos\theta = \cos\widehat{\rho\,.\,\varphi\rho} = -\frac{S\rho\varphi\rho}{T\rho\,.\,T\varphi\rho};$$

and therefore

$$\cos\theta = \frac{a_{11}x^2 + xy\,(a_{12} + a_{21}) + y^2 a_{22}}{\sqrt{x^2 + y^2}\,\sqrt{(a_{11}x + a_{12}y)^2 + (a_{21}x + a_{22}y)^2}} .$$

2. If ρ is unchanged in direction by φ we must have $V\rho\varphi\rho = 0$ or $\varphi\rho = \lambda\rho$, from which we get

$$\lambda x = a_{11}x + a_{12}y\,, \quad \lambda y = a_{21}x + a_{22}y\,;$$

or

$$(a_{11} - \lambda)\,x + a_{12}y = 0\,, \quad a_{21}x + (a_{22} - \lambda)\,y = 0\,.$$

* See Taber, American Journal of Mathematics, Vol. XII.

These equations are satisfied by values of x and y other than zero when

$$\Delta' = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0\,;$$

and therefore

$$\lambda = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4\Delta}}{2}\,,$$

where $\Delta = \left| \varphi \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Denoting these two values of λ, which are the latent roots of, φ by g_1 and g_2 we have

$$\frac{x}{g_1 - a_{22}} = \frac{y}{a_{21}} = m\,, \quad \text{or} \quad \frac{x}{a_{12}} = \frac{y}{g_1 - a_{11}} = n\,;$$

and

$$\frac{x}{g_2 - a_{22}} = \frac{y}{a_{21}} = m\,, \quad \text{or} \quad \frac{x}{a_{12}} = \frac{y}{g_2 - a_{11}} = n\,.$$

The two vectors whose directions are unchanged by φ are therefore

$$\rho_1 = m\,\{(g_1 - a_{22})\,i + a_{21}j\} = n\,\{a_{12}i + (g_1 - a_{11})j\}\,,$$

$$\rho_2 = m\,\{(g_2 - a_{22})\,i + a_{21}j\} = n\,\{a_{12}i + (g_2 - a_{11})j\}\,.$$

These vectors are real when g_1 and g_2 are real and it is obvious that ρ_1 and ρ_2 will coincide when $g_1 = g_2$.

Operating with φ on ρ_1 and ρ_2, we get

$$\varphi\rho_1 = m\,[\,\{a_{11}(g_1 - a_{22}) + a_{12}a_{21}\}\,i + \{a_{21}(g_1 - a_{22}) + a_{21}a_{22}\}\,j] = g_1\rho_1\,;$$

similarly

$$\varphi\rho_2 = g_2\rho_2$$

Again,

$$\cos \widehat{\rho_1 \cdot \rho_2} = \frac{(g_1 - a_{22})\,(g_2 - a_{22}) + a_{21}^{\,2}}{\sqrt{(g_1 - a_{22})^2 + a_{21}^{\,2}} \cdot \sqrt{(g_2 - a_{22})^2 + a_{21}^{\,2}}}$$

$$= \frac{a_{21}\,(a_{12} - a_{21})}{\sqrt{(g_1 - a_{22})^2 + a_{21}^{\,2}} \cdot \sqrt{(g_2 - a_{22})^2 + a_{21}^{\,2}}}$$

$$= 0\,, \text{ if } \varphi \text{ is symmetric.}$$

If φ leaves all vectors unchanged in direction the equation

$$\lambda x = a_{11}x + a_{12}y\,, \quad \lambda y = a_{21}x + a_{22}y$$

must be independent of x and y, and therefore $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = \lambda$.

Consequently

$$\varphi = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and is a multiple of unity.

3. If φ transforms the vector ρ perpendicular to itself we must have $\cos \widehat{\rho \, . \, \varphi\rho} = 0$, that is

$$a_{11}x^2 + (a_{12} + a_{21})xy + a_{22}y^2 = 0 \, ;$$

and $\therefore$

$$\frac{x}{y} = \frac{-a_{12} - a_{21} \pm \sqrt{(a_{12} + a_{21})^2 - 4\, a_{11}a_{22}}}{2\, a_{11}}$$

$$= \frac{-a_{12} - a_{21} \pm \sqrt{(a_{12} - a_{21})^2 - 4\Delta}}{2\, a_{11}} .$$

Then

$$\rho_3 = y \left\{ \left[\frac{-a_{12} - a_{21} + \sqrt{(a_{12} - a_{21})^2 - 4\Delta}}{2\, a_{11}} \right] i + j \right\} ,$$

$$\rho_4 = y \left\{ \left[\frac{-a_{12} - a_{21} - \sqrt{(a_{12} - a_{21})^2 - 4\Delta}}{2\, a_{11}} \right] i + j \right\}$$

are two vectors transformed by φ perpendicular to themselves. They are real or imaginary according as $(a_{12} + a_{21})^2 \geqq 4a_{11}a_{22}$ or $(a_{12} + a_{21})^2 < 4a_{11}a_{22}$, respectively.

They are at right angles to each other if $a_{11} + a_{22} = 0$, which is the condition that φ is a vector.*

If φ transforms all vectors perpendicular to themselves then

$$a_{11}x^2 + (a_{12} + a_{21})xy + y^2a_{22} = 0$$

independently of x and y;

$$\therefore \quad a_{11} = a_{22} = 0 \, , \quad a_{12} = -a_{21} \, ;$$

and φ is skew symmetric. That is a skew symmetric matrix rotates all vectors through 90° and increases their length a_{12} times.

4. A pure or non-rational strain consists in altering the lengths of two lines at right angles to each other without altering their directions.

We have seen in Art. 2 that the two vectors ρ_1 and ρ_2 are at right angles to each other when φ is symmetric (self-conjugate), which is the condition, therefore, for a pure strain.

5. For a pure rotation,

$$T\varphi\rho = T\rho \, , \text{ and } \cos \widehat{\rho \, . \, \sigma} = \cos \widehat{\varphi\rho \, . \, \varphi\sigma} \, .$$

* See paper by Prof. Cayley, Messenger of Mathematics, Vol. 14, p. 146.

From either of these we get

$$\sqrt{x^2 + y^2} = \sqrt{(a_{11}x + a_{12}y)^2 + (a_{21}x + a_{22}y)^2}\,;$$

$$\therefore\quad a_{12}^2 + a_{22}^2 = 1\,,\quad a_{11}^2 + a_{21}^2 = 1\,,\quad a_{11}a_{12} + a_{21}a_{22} = 0\,;$$

from which we may derive

$$a_{11}^2 + a_{12}^2 = 1\,,\quad a_{21}^2 + a_{22}^2 = 1\,,\quad a_{11}a_{21} + a_{12}a_{22} = 0\,;$$

and

$$\therefore\quad a_{12} = \pm a_{21}\,,\quad a_{11} = \mp a_{22}\,,$$

and φ is orthogonal.

Again, $\cos \widehat{\rho\,.\,\varphi\rho} = \dfrac{a_{11}x^2 + xy\,(a_{12} + a_{21}) + a_{22}y^2}{x^2 + y^2}$, which must be the same for all vectors ρ, and consequently independent of x and y;

$$\therefore\quad a_{11} = a_{22}\,,\quad a_{12} = -a_{21}\,,\quad \text{and}\quad \cos \widehat{\rho\,.\,\varphi\rho} = a_{11}\,.$$

The direction of rotation is given by the sign of a_{12}; if it is positive the rotation is negative, and if it is negative the rotation is positive.*

We may now write

$$\varphi = \begin{pmatrix} a_{11} & \pm\sqrt{1 - a_{11}^2} \\ \mp\sqrt{1 - a_{11}^2} & a_{11} \end{pmatrix},\ \text{or}\ \varphi = \begin{pmatrix} -a_{11} & \pm\sqrt{1 - a_{11}^2} \\ \mp\sqrt{1 - a_{11}^2} & -a_{11} \end{pmatrix}.$$

The latent roots become

$$g_1 = a_{11} + \sqrt{a_{11}^2 - 1}\,,\quad g_2 = a_{11} - \sqrt{a_{11}^2 - 1}\,;$$

both of which are imaginary unless $a_{11}^2 - 1 = 0$, in which case $g_1 = g_2 = \pm 1$. If g_1 and g_2 are imaginary, then obviously ρ_1 and ρ_2 are imaginary; otherwise they are zero.

6. Any matrix φ can be written in the form $\varphi = \varphi_1 + \varphi_2$, where φ_1 is symmetric and φ_2 is skew symmetric; from which we see that any homogeneous strain is equivalent to a pure strain plus a rotation through 90°, accompanied by a uniform dilatation measured by c_{12}, where $c_{12}^2 = |\varphi_2|$.

Again any matrix φ can be written in the form $\varphi = \varphi_1 + \varphi_2$, where φ_1 is a symmetric and φ_2 is a skew matrix.

Let

$$\varphi_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \quad\text{and}\quad \varphi_2 = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12} & c_{11} \end{pmatrix},$$

then $\varphi = \varphi_1 + \varphi_2$ shows that any homogeneous strain is equivalent to a pure

* By a_{12} here is meant the constituent of φ in the first row and second column.

strain plus a rotation through an angle $\cos^{-1}\dfrac{c_{11}}{\sqrt{c_{11}^2 + c_{12}^2}}$, accompanied by a uniform dilatation measured by $\sqrt{c_{11}^2 + c_{12}^2}$.

If $c_{11}^2 + c_{12}^2 = 1$, then φ_2 is orthogonal, and represents a pure rotation. In this case, to express the constituents of φ_1 and φ_2 in terms of those of φ we have

$$a_{11} = b_{11} + c_{11}, \quad a_{12} = b_{12} + \sqrt{1 - c_{11}^2}, \quad a_{21} = b_{12} - \sqrt{1 - c_{11}^2}, \quad a_{22} = b_{22} + c_{11};$$

$$\therefore \quad b_{12} = \tfrac{1}{2}(a_{12} + a_{21}), \quad c_{11} = \pm \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2},$$

$$b_{11} = a_{11} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2}, \quad b_{22} = a_{22} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2};$$

and $\therefore$

$$\varphi_1 = \begin{pmatrix} a_{11} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} & \tfrac{1}{2}(a_{12} + a_{21}) \\ \tfrac{1}{2}(a_{12} + a_{21}) & a_{22} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} \end{pmatrix},$$

and

$$\varphi_2 = \begin{pmatrix} \pm \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} & \tfrac{1}{2}(a_{12} - a_{21}) \\ -\tfrac{1}{2}(a_{12} - a_{21}) & \pm \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} \end{pmatrix};$$

which shows that neither φ_1 nor φ_2 will be real unless $a_{12} - a_{21} \leqq 2$.

7. If $\varphi = \varphi_1\varphi_2$ where φ_1 is symmetric and φ_2 is orthogonal to express the constituents of φ_1 and φ_2 in terms of those of φ.

From the equation

$$\varphi = \varphi_1\varphi_2$$

we have

$$a_{11} = b_{11}c_{11} - b_{12}\sqrt{1 - c_{11}^2}, \quad a_{12} = b_{11}\sqrt{1 - c_{11}^2} + b_{12}c_{11},$$

$$a_{21} = c_{11}b_{12} - b_{22}\sqrt{1 - c_{11}^2}, \quad a_{22} = b_{12}\sqrt{1 - c_{11}^2} + b_{22}c_{11};$$

$$\therefore \quad c_{11} = \frac{a_{11} + a_{22}}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$\sqrt{1 - c_{11}^2} = \frac{a_{12} - a_{21}}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$b_{11} = \frac{a_{11}(a_{11} + a_{22}) + a_{12}(a_{12} - a_{21})}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$b_{12} = \frac{a_{12}a_{22} + a_{11}a_{21}}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$b_{22} = \frac{a_{22}(a_{11} + a_{22}) - a_{21}(a_{12} - a_{21})}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}}.$$

In this case both φ_1 and φ_2 are real, if φ is real.

8. The matrix $\phi = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ rotates all vectors into the direction of $\sigma = i + j$.

If
$$\rho = xi + yj ,$$
then
$$\phi\rho = a(x+y)(i+j) = a(x+y)\sigma ,$$
$$\phi^2\rho = 2a^2(x+y)\sigma ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\phi^n\rho = 2^{n-1} \cdot a^n \cdot (x+y)\sigma .$$

Part II.—Space of Three Dimensions.

9. Let $\phi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, and $\rho = xi + yj + zk$;

then
$$\phi\rho = (a_{11}x + a_{12}y + a_{13}z)\,i + (a_{21}x + a_{22}y + a_{23}z)\,j + (a_{31}x + a_{32}y + a_{33}z)\,k ,$$
and

$\cos \widehat{\rho \,.\, \phi\rho} =$
$$\frac{x^2a_{11} + xy(a_{12} + a_{21}) + y^2a_{22} + xz(a_{13} + a_{31}) + z^2a_{33} + yz(a_{23} + a_{32})}{\sqrt{x^2 + y^2 + z^2}\,\sqrt{(a_{11}x + a_{12}y + a_{13}z)^2 + (a_{21}x + a_{22}y + a_{23}z)^2 + (a_{31}x + a_{32}y + a_{33}z)^2}} .$$

The vectors unchanged by strain are given by $V\rho\phi\rho = 0$ or $\phi\rho = \lambda\rho$. From these we get
$$a_{11}x + a_{12}y + a_{13}z = \lambda x ,$$
$$a_{21}x + a_{22}y + a_{23}z = \lambda y ,$$
$$a_{31}x + a_{32}y + a_{33}z = \lambda z .$$

There are three values of λ, given by $\Delta' = 0$, for which these equations are satisfied by values of x, y, and z other than zero, where
$$\Delta' = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} .$$

Then

$$\frac{x}{A'_{11}} = \frac{y}{A'_{12}} = \frac{z}{A'_{13}} = l\,,$$

$$\frac{x}{A'_{21}} = \frac{y}{A'_{22}} = \frac{z}{A'_{23}} = m\,;$$

or

$$\frac{x}{A'_{31}} = \frac{y}{A'_{32}} = \frac{z}{A'_{33}} = n\,;$$

where

$$A'_{11} = \begin{vmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{vmatrix}\,, \quad A'_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} - \lambda \end{vmatrix}\,, \text{ etc.}$$

Denoting the latent roots of φ by g_1, g_2, g_3, we have

$$\begin{aligned} \rho_1 &= l\left[\{g_1^2 - g_1(a_{22} + a_{33}) + A_{11}\}\, i + \{g_1 a_{21} + A_{12}\}\, j + \{g_1 a_{31} + A_{13}\}\, k\right] \\ &= m\left[\{g_1 a_{12} + A_{21}\}\, i + \{g_1^2 - g_1(a_{11} + a_{33}) + A_{22}\}\, j + \{g_1 a_{32} + A_{23}\}\, k\right] \\ &= n\left[\{g_1 a_{13} + A_{31}\}\, i + \{g_1 a_{23} + A_{32}\}\, j + \{g_1^2 - g_1(a_{11} + a_{22}) + A_{33}\}\, k\right], \\ \rho_2 &= l\left[\{g_2^2 - g_2(a_{22} + a_{33}) + A_{11}\}\, i + \{g_2 a_{21} + A_{12}\}\, j + \{g_2 a_{31} + A_{13}\}\, k\right] = \text{etc.}, \\ \rho_3 &= l\left[\{g_3^2 - g_3(a_{22} + a_{33}) + A_{11}\}\, i + \{g_3 a_{21} + A_{12}\}\, j + \{g_3 a_{31} + A_{13}\}\, k\right] = \text{etc.} \end{aligned}$$

There are therefore three vectors whose directions are unchanged by the strain, all of which are real if g_1, g_2, and g_3 are real. One at least of these vectors is real, since one of the latent roots must be real.

Operating on ρ_1, ρ_2, ρ_3 by φ, we get

$$\varphi\rho_1 = g_1\rho_1\,, \quad \varphi\rho_2 = g_2\rho_2\,, \quad \varphi\rho_3 = g_3\rho_3\,.$$

10. If φ is symmetric, then g_1, g_2, g_3 are real, ρ_1, ρ_2, ρ_3 form a rectangular system, and the strain is pure. The equations $\varphi\rho_1 = g_1\rho_1, \varphi\rho_2 = g_2\rho_2, \varphi\rho_3 = g_3\rho_3$ show that for true physical pure strain g_1, g_2, g_3, besides being real, must be positive.*

11. For a pure rotation we must have

$$T\varphi\rho = T\rho\,, \quad \cos\widehat{\rho\,.\,\sigma} = \cos\widehat{\varphi\rho\,.\,\varphi\sigma}\,;$$

$$\begin{aligned} \therefore\quad a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1\,, & a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0\,, \\ a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1\,, & a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0\,, \\ a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1\,, & a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0\,. \end{aligned}$$

* Kelland and Tait, 2d Ed., Chap. X, Sect. V.

From these we may obtain

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1\,, \qquad a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0\,,$$

$$a_{21}^2 + a_{22}^2 + a_{23}^2 = 1\,, \qquad a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0\,,$$

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1\,, \qquad a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0\,.$$

The matrix φ is then orthogonal and $\therefore a_{11} = A_{11}$, and in general, $a_{rs} = A_{rs}$.*
One of the latent roots is real and equal to ± 1,* suppose it to be g_1. Then

$$\rho_1 = t\,\{(2 \pm (g_2 + g_3 + a_{11}) + A_{11})\,i + (A_{12} \pm a_{21})\,j + (A_{13} \pm a_{31})\,k\}\,.$$

If all the roots are real, then φ is symmetric, and $g_2 = \pm 1$, $g_3 = \pm 1$.* If $g_1 = g_2 = g_3 = 1$, there is obviously no rotation, and if $g_1 = g_2 = g_3 = -1$, then the rotation is improper or physically impossible.

To find the amount of rotation, we find a vector ρ_4 perpendicular to ρ_1, the axis of rotation, and then the angle θ of rotation is given by

$$\cos\theta = \cos \widehat{\rho_4 \cdot \varphi\rho_4}\,.$$

The vector $\rho_4 = (a_{11} - 1)\,i + a_{12}\,j + a_{13}k$ is easily seen to be perpendicular to ρ_1.

Then
$$\varphi\rho_4 = (1 - a_{11})\,i - a_{21}\,j - a_{31}k\,,$$
and

$$\cos\theta = \frac{-(a_{11}-1)^2 - a_{21}a_{12} - a_{13}a_{31}}{\sqrt{(a_{11}-1)^2 + a_{12}^2 + a_{13}^2}\,\sqrt{(a_{11}-1)^2 + a_{21}^2 + a_{31}^2}}$$

$$= \tfrac{1}{2}\,(-a_{11}^2 + 2a_{11} - 1 - a_{11}a_{33} - a_{11}a_{22} + A_{33} + A_{22})\,(1 - a_{11})^{-1}$$

$$= \tfrac{1}{2}\,[(a_{11} + a_{22} + a_{33}) - 1]$$

$$= \tfrac{1}{2}\,(g_1 + g_2 + g_3 - 1)\,,\dagger$$

$$\cos 2\theta = \tfrac{1}{2}\,(g_1^2 + g_2^2 + g_3^2 - 1)\,,$$

and generally

$$\cos n\theta = \tfrac{1}{2}\,(g_1^n + g_2^n + g_3^n - 1)\,.$$

12. Suppose φ is a skew matrix of the form

$$\varphi = \left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{11} & a_{23} \\ -a_{13} & -a_{23} & a_{11} \end{array}\right).$$

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* See my paper, American Journal of Mathematics, Vol. XV, No. 3, § 4.

† See Routh, Rigid Dynamics, 3d Ed., Chap. V, Art. 184.

One of the latent roots (g_1 say) will be real and equal to a_{11}, while the remaining two will be complex imaginary. The vectors ρ_2 and ρ_3 will be imaginary while $\rho_1 = l a_{23}(a_{23} i - a_{13} j + a_{12} k)$ and $\varphi \rho_1 = a_{11} \rho_1$.

The vector $\rho_4 = (a^{13} - a_{12})\, i - (a_{12} - a_{23})\, j + (a_{23} - a_{13}) k$ is readily seen to be perpendicular to ρ_1.

Then

$$\begin{aligned} \varphi \rho_4 = &\{ a_{11}(a_{13} - a_{12}) - a_{12}(a_{12} - a_{23}) + a_{13}(a_{23} - a_{13}) \}\, i \\ &+ \{ - a_{12}(a_{13} - a_{12}) - a_{11}(a_{12} - a_{23}) + a_{23}(a_{23} - a_{13}) \}\, j \\ &+ \{ - a_{13}(a_{13} - a_{12}) + a_{23}(a_{12} - a_{23}) + a_{11}(a_{23} - a_{13}) \}\, k, \end{aligned}$$

and

$$\cos \widehat{\rho_4 \cdot \varphi \rho_4} = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2}};$$

which is seen to be the same for all vectors perpendicular to ρ_1.

We also have

$$T \varphi \rho_4 = \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2} \,.\, T \rho_4,$$

which shows that there is a uniform dilatation perpendicular to the axis of rotation measured by $\sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2}$.

13. If φ is skew symmetric, then $a_{11} = 0$, and therefore $\cos \widehat{\rho_4 \cdot \varphi \rho_4} = 0$; that is, φ rotates all vectors perpendicular to ρ_1 through 90°, and uniformly elongates them $\sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2}$ times.

Postscript.—On obtaining the above results for a skew symmetric matrix I observed that they were at variance with those given in Tait's Quarternions, 3d Ed., Chap. XI, Art. 381, pp. 298. I sent my solution to Prof. Tait, who replied that the fallacy was in his book, and not in my work. He was evidently thinking of another case when giving the results for this.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, *January* 13, 1894.

SOLUTIONS OF EXERCISES.

90

Show that if x be acute, one value of $\log(1 + \cos 2x + i \sin 2x)$ is $ix + \log(2 \cos x)$.

SOLUTION.

Use the identity

$$\log(A + iB) = \tfrac{1}{2}\log(A^2 + B^2) + i \tan^{-1}\frac{B}{A},$$

$$\begin{aligned}\therefore\ \log(1 + \cos 2x + i \sin 2x) &= \tfrac{1}{2}\log\{(1 + \cos 2x)^2 + (\sin 2x)^2\} \\ &\quad + i \tan^{-1}\frac{\sin 2x}{1 + \cos 2x} \\ &= \log(2 \cos x) + ix.\end{aligned}$$

If $x < \frac{1}{2}\pi$, then $\log(2 \cos a)$ is real; but if $x > \frac{1}{2}\pi$, then $\log(2 \cos x)$ may be replaced by $\log[2 \sin(x - \frac{1}{2}\pi)] + i(2n + 1)\pi$. [*Jesse Pawling, Jr.*]

242

The curve $\tan x + \tan y = a$ is symmetrical with regard to certain lines parallel to $x + y = 0$.

SOLUTION.

Turning the axes through 45°, the equation becomes

$$\tan\frac{x - y}{\sqrt{2}} + \tan\frac{x + y}{\sqrt{2}} = a,$$

i. e. $$2\left[1 + \tan^2\frac{y}{\sqrt{2}}\right]\tan\frac{x}{\sqrt{2}} = a\left[1 - \tan^2\frac{x}{\sqrt{2}}\tan^2\frac{y}{\sqrt{2}}\right];$$

and the curve is now to be proved symmetrical with regard to certain lines $x = h$. To transfer the origin to $(h, 0)$, put $x + h$ for x; and write for brevity $\tan\frac{x}{\sqrt{2}} = \alpha$, $\tan\frac{y}{\sqrt{2}} = \beta$, $\tan\frac{h}{\sqrt{2}} = \gamma$; then the equation becomes

$$2(1 + \beta^2)(\alpha + \gamma)(1 - \alpha\gamma) = a(1 - \alpha\gamma)^2 - a\beta^2(\alpha + \gamma)^2.$$

The axis of y is an axis of symmetry if this equation be free from odd powers of α, i. e. if

$$2(1 + \beta^2)(1 - \gamma^2) = -2a(\gamma + \beta^2\gamma);$$

hence h is to be found from the equation

$$\frac{2\gamma}{1-\gamma^2} = -\frac{2}{a},$$

giving

$$\tan 2\left(\frac{h}{\sqrt{2}}\right) = -\frac{2}{a}, \quad \text{or} \quad -h\sqrt{2} = \cot^{-1}\tfrac{1}{2}a\,.$$

It follows that the given curve is symmetrical with regard to any of the lines represented by the equation $x + y + \cot^{-1}\frac{a}{2} = 0$, referred to the original axes. These lines are equivalent at intervals of $\frac{\pi}{\sqrt{2}}$ units of length.

[*C. D. Child.*]

294

FIND the equation, on the developed surface of a cone, of the section line made by the intersection of a plane with the cone.

SOLUTION.

Let the equations of the cone and plane be

$$x^2 + y^2 = z^2 \tan^2 a\,, \tag{1}$$

$$x\cos\beta + z\sin\beta = p\,, \tag{2}$$

and let φ be the azimuth of the generator passing through (x, y, z), i. e. the angle which the plane of the generator and the axis of z, makes with the plane (xz); then when the cone is developed, the angle between this generator and the generator of zero azimuth, is easily seen to be $\varphi \sin a$; and the polar equation of the section line may be obtained by eliminating x, y, z, φ, from (1), (2) above, and (3), (4), (5) following:

$$\theta = \varphi \sin a\,, \tag{3}$$

$$y = x \tan\varphi\,, \tag{4}$$

$$r^2 = x^2 + y^2 + z^2\,, \tag{5}$$

giving for the required equation

$$r\,[\tan\beta + \tan a \cos(\theta \operatorname{cosec} a)] = p \sec a \sec\beta\,.$$

The form of the curve depends on the relative values of a and β; for instance, if $\tan\beta$ is numerically less than $\tan a$, there are two values of θ (between the extreme limits 0 and $2\pi \sin a$ for the developed surface) that give infinite values to r.

If the curve be extended beyond these limits, it will repeat itself, since r is a periodic function of θ, having the period $2\pi \sin \alpha$; and the waves will overlap unless cosec α is an integer. In the latter case the curve is algebraic, and of the degree 2 cosec α; the numbers of maxima and minima of the radii vectores being each equal to cosec α, when the value of β is such that the curve is closed. When cosec α is m/n, an improper fraction in its lowest terms, the degree of the curve is $2m$. [*James McMahon.*]

355

Required the locus of the point in the normal to a conic, which is equally distant from the focus and the foot of the normal. [*Geo. R. Dean.*]

CORRECTED SOLUTION.

Taking the origin at the center of the ellipse we have for the equation of the normal

$$y - b \sin \varphi = \frac{a}{b} \tan \varphi \, (x - a \cos \varphi),$$

or

$$y = \frac{a}{b} \tan \varphi - \frac{a^2 - b^2}{b} \sin \varphi .$$

The locus required is the intersection of this normal with the line

$$y = -\frac{a}{b} \tan \varphi \, (x - ae) .$$

Eliminating φ, we get

$$y = 0 ,$$

and

$$\{(x - ae + 1)^2 - a^2 e^4\} (x - ae)^2 + (1 - e^2) \, y^2 (x - ae + 1)^2 = 0 ,$$

the x-axis, which is excluded, and a curve having $(x - ae + 1)^2 = 0$ for the equation of two coincident asymptotes. The curve cuts the x-axis at the points $x = ae\,(1 \pm e) - 1$, and the focus is a conjugate point. [*Geo. R. Dean.*]

359

Four normals can be drawn from a point to a limaçon; if the feet of two of the normals lie on a line through the node, the feet of the other two lie on a line through the focus. [*Frank Morley.*]

SOLUTION.

The limaçon, regarded as traced by a point attached to a circle which rolls on an equal circle, is expressed at once by the equation

$$x = 2at - \beta t^2 ,$$

where α, β are real, and $t = 1$. Writing y for the conjugate of x, the normal is

$$x(\alpha t - \beta) - y t^3 (\alpha - \beta t) = \alpha\beta(t^3 - t)\,.$$

Let the roots of this equation in t be t_1, t_2, t_3, t_4; then the conditions that the normals at these points meet are

$$\Sigma t_1 t_2 = 0\,, \quad \Sigma t_1 + \Sigma t_1 t_2 t_3 = (1 + t_1 t_2 t_3 t_4)\,\alpha/\beta\,.$$

If t_3, t_4 are ends of a nodal chord, then $t_3 = -t_4$, and therefore

$$t_1 t_2 = -t_3 t_4\,,$$

and

$$(t_1 + t_2)(1 + t_3 t_4) = (1 - t_3^2 t_4^2)\,\alpha/\beta\,,$$

so that

$$t_1 + t_2 = (1 + t_1 t_2)\,\alpha/\beta\,.$$

Hence

$$(\alpha - \beta t_1)(\alpha - \beta/t_1) = -t_1 t_2 = (\alpha - \beta t_2)/(\alpha - \beta/t_2)\,.$$

Hence the amplitudes of $(\alpha - \beta t_1)^2$ and $(\alpha - \beta t_2)^2$ are equal. Now the focus is α^2/β; and the stroke from the focus to any point t is $-(\alpha - \beta t)^2/\beta$; and the statement is proved. [*Frank Morley.*]

360

Normals at the ends of a nodal chord of a given limaçon mark off an involution on the axis of the curve. [*Frank Morley.*]

SOLUTION.

For the notation see solution of Ex. 359. The normal at t_1 meets the real axis at ν_1, where

$$\nu_1 \{\alpha t_1 - \beta(1 + t_1^2)\} = -\alpha\beta t_1\,,$$

or

$$1/\nu_1 + 1/\beta = (t_1 + t_1^{-1})/\alpha\,.$$

Therefore, when $t_1 + t_2 = 0$,

$$1/\nu_1 + 1/\nu_2 = -2/\beta\,.$$

That is, the points are harmonic with the fixed points 0 and $-\beta$.

[*Frank Morley.*]

361

Let r be the base of a system of numeration. Find the condition that in the quotient of the number

$$A = aaa \ldots a \qquad (r - 1 \text{ places})$$

divided by $r - 1$, there shall appear all but one of the digits of the system (0 excluded), and determine the lacking digit. [*Edgar H. Johnson.*]

SOLUTION.

If 0 appear in this quotient the division must be exact before the end is reached. In order that any number be exactly divisible by $r - 1$, it is necessary and sufficient that the sum of its digits be so divisible; so that if 0 appears, a and $r - 1$ must have a common divisor, and conversely. By application of the same principle it is easily seen that this is also the necessary and sufficient condition that in this quotient two of the digits be the same. So that if, and only if, a and $r - 1$ are relative primes, none of the $r - 2$ digits in the quotient will be the same and one digit besides 0 will be lacking.

If $r - 1$ is prime every digit, of course, satisfies the required condition,

$$r^{r-2} + r^{r-3} + \ldots r + 1/r - 1 = r^{r-3} + 2\,r^{r-4} + 3\,r^{r-5} + \ldots (r-3)\,r + (r-1)\,;$$

So that the quotient of the number

$$111 \ldots 1 \qquad [r - 1 \text{ places}]$$

divided by $r - 1$ is the number

$$N = 123 \ldots r-4\ r-3\ r-1\,.$$

If from the dividend we subtract the quotient N, the remainder

$$r-1\ r-2\ r-3 \ldots 432$$

is the quotient of the number

$$r-2\ r-2\ r-2 \ldots r-2 \qquad [r - 1 \text{ places}]$$

divided by $r - 1$.

To find the quotient of A divided by $r - 1$ we have only to multiply N by a.

The sum of all the digits in the system is $\frac{r(r-1)}{2}$. The sum of the digits in N is $\frac{r^2 - 3r + 4}{2}$.

The remainder after the division of any number by $r - 1$ is the same as after division of the sum of its digits by $r - 1$. Applying this to N we see that the remainder is 1 if r is even and $\frac{r+1}{2}$ if r is odd.

When r is even the sum of the digits in aN is congruent to a and the sum of all the digits in the system is congruent to 0. Hence the lacking digit in aN is $r - 1 - a$.

When r is odd the sum of the digits in aN is congruent to $\frac{a(r+1)}{2}$; and the sum of all the digits in the system is congruent to $\frac{r-1}{2}$. The miss-

ing digit is, as before, $r - 1 - a$; for

$$r - 1 - a + \frac{a(r+1)}{2} \equiv \frac{a(r-1)}{2} \equiv \frac{r-1}{2} \pmod{r-1},$$

it being noticed that $\frac{(a-1)(r-1)}{2} \equiv 0 \pmod{r-1}$, because a is odd, being relative prime to $r-1$. [*Edgar H. Johnson.*]

364

FIND two complete integrals of the equation

$$\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 = \frac{x-y}{z}.$$ [*Geo. R. Dean.*]

SOLUTION.

Writing the equation in the form $z(p^2 + q^2) - (x - y) = 0$, and using Charpit's method, we have the subsidiary equations,

$$\frac{dx}{2zp} = \frac{dy}{2zq} = \frac{dz}{2z(p^2+q^2)} = \frac{-dp}{-1 + p(p^2+q^2)} = \frac{-dq}{1 + q(p^2+q^2)}.$$

1. From the relations between dz, dp and dq, we get

$$\frac{dz}{2z} = \frac{-d(p+q)}{p+q};$$

whence

$$(p+q)^2 = \frac{c}{z}.$$

Combining this equation with the original, and substituting the values of p and q in $dz = pdx + qdy$, we have

$$2z^{\frac{1}{2}}dz = [c^{\frac{1}{2}} + \{2(x-y) + c\}^{\frac{1}{2}}]\,dx + [c^{\frac{1}{2}} - \{2(x-y) + c\}^{\frac{1}{2}}]\,dy,$$

the integral of which is

$$\tfrac{4}{3}z^{\frac{3}{2}} = c^{\frac{1}{2}}(x+y) + \tfrac{1}{3}\{2(x-y) - c\}^{\frac{3}{2}}.$$

2. From the relations between dx, dz, and dp we get

$$p = \left[\frac{x-c}{z}\right]^{\frac{1}{2}}, \qquad q = \left[\frac{c-y}{z}\right]^{\frac{1}{2}},$$

which substituted in $dz = pdx + qdy$, give after integration

$$z^{\frac{3}{2}} = (x-c)^{\frac{3}{2}} - (c-y)^{\frac{3}{2}} + c.$$ [*Geo. R. Dean.*]

365

Show that

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

is a surface of revolution, and find its axis. [*Geo. R. Dean.*]

SOLUTION.

The differential equation of a surface of revolution is

$$(ly - mx) + (ny - mz)\,p + (lz - mx)\,q = 0\,,$$

where l, m, and n are the direction-cosines of the axis, and

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial x}{\partial y}.$$

The values of p and q for the given equation give, when substituted in the differential equation,

$$(ly - mx)(z^2 - xy) + (ny - mz)(x^2 - yz) + (lz - nx)(y^2 - xz) = 0\,,$$

which is satisfied when $l = m = n$. Hence the equations of the axis are

$$x = y = z\,.$$

[*Geo. R. Dean.*]

366

Show that for any parabola $y = x^2 + ax + b$, the area included between the curve, any two ordinates, and the x-axis, is equal to the product of the ordinate midway between the bounding ordinates and the interval between them, plus one-twelfth the cube of this interval. [*W. H. Echols.*]

SOLUTION.

While many simple solutions may be given, the following is not without interest as an application of the general formula (23), Annals of Mathematics, Vol. VII, p. 21.*

Let

$$f(x) = x^2 + ax + b\,.$$

We have

$$\int_q^p f(x)\,dx = (p - q)\,f(a) + Af'(\beta) + Bf''(\gamma)\,,$$

wherein a, β, γ are arbitrary, and

$$A = \frac{1}{2\,!}(p^2 - q^2) - a(p - q)\,,$$

$$B = \frac{1}{3\,!}(p^3 - q^3) + \frac{1}{2\,!}\beta(p^2 - q^2) - a\beta(p - q) + \frac{1}{2\,!}a^2(p - q)\,.$$

Make $a = \frac{1}{2}(p + q)$, then $A = 0$ and $B = \frac{1}{24}(p - q)^3$, while $f''(\gamma) = 2$.

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* Otherwise, immediately by inspection of formulæ (45), p. 33 ; (63), (64), p. 36 ; (68), p. 37.

Whence

$$\int_q^p f(x)dx = (p-q)f\left[\frac{p+q}{2}\right] + \tfrac{1}{12}(p-q)^3.$$

[*W. H. Echols.*]

Also solved by Messrs. Geo. R. Dean, Eric Doolittle, and H. Y. Benedict.

EXERCISES.

369

SOLVE the equation

$$\frac{d^2y}{dx^2} - y = e^{\frac{1}{2}x^2}.$$

[*Geo. R. Dean.*]

370

FROM a point A on the equator a northeast rhumb line is drawn; find at what latitude it again strikes the meridian of A, and express the length of the rhumb line in radii. [*James McMahon.*]

371

PROVE that the ratio of

$$\begin{vmatrix} \sigma_1(2u_1), & \sigma_1(2u_2), & \sigma_1(2u_3) \\ \sigma_2(2u_1), & \sigma_2(2u_2), & \sigma_2(2u_3) \\ \sigma_3(2u_1), & \sigma_3(2u_2), & \sigma_3(2u_3) \end{vmatrix}$$

to

$$\sigma(u_2+u_3)\,\sigma(u_3+u_1)\,\sigma(u_1+u_2)\,\sigma(u_2-u_3)\,\sigma(u_3-u_1)\,\sigma(u_1-u_2)$$

is independent of the arguments u_λ; and that its value is

$$4(e_2-e_3)(e_3-e_1)(e_1-e_2);$$

the notation being that of Weierstrass. [*Frank Morley.*]

372

WHEN the bilinear invariant of two binary n-ics is zero, we say that the n-ics are *apolar*. When also the n-ics coincide we say that either is self-apolar. And we may apply the same adjectives to the sets of n points (or n-*ads*) which represent the zeros of the n-ics. Any odd set of points is, we know, self-apolar (Salmon's Higher Algebra, § 153). Prove that an even set is self-apolar when the first polar of any point of the set, with regard to the rest, is self-apolar. [*Frank Morley.*]